PHYS 561 HOMEWORK COLLECTION

Problem 1 A lightweight pole 20m long lies on the ground next to a barn 15m long. An Olympic athlete picks up the pole, carries it far away, and runs with it toward the end of the barn at a speed 0.8c. His friend remains at rest, standing by the door of the barn. Attempt all parts of this question, even if you can't answer some.

(a) How long does this friend measure the pole to be, as it approaches the barn?

(b) The barn door is initially open, and immediately after the runner and pole are entirely inside the barn, the friend shuts the door. How long after the door is shut does the front of the pole hit the other end of the barn, as measured by the friend? Compute the interval between the events of shutting the door and hitting the wall. Is it spacelike, timelike, or null?

(c) In the reference frame of the runner, what is the length of the barn and the pole?

(d) Does the runner believe that the pole is entirely inside the barn when its front hits the end of the barn? Can you explain why?

(e) After the collision, the pole and the runner come to rest relative to the barn. From the friend's point of view, the 20m pole is now inside a 15m barn, since the barn door was shut before the pole stopped. How is this possible? Alternatively, from the runner's point of view, the collision should have stopped the pole *before* the door closed, so the door could not be closed at all. Was or was not the door closed with the pole inside?

(f) Draw a spacetime diagram from the friend's point of view and use it to illustrate and justify all your conclusions.

Problem 2 The 4-velocity u corresponds to 3-velocity $\underline{v}$. Express:

(a) u^0 in terms of $|\underline{v}|$

(b) u^β (β= 1,2,3)terms of $\underline{v}$

(c) u^0 in terms of u^β

(d) $d/d\tau$ in terms of d/dt and $\underline{v}$

(e) v^β in terms of u^β

(f) $|\underline{v}|$ in terms of u^0

Problem 3 Find the matrix for the Lorentz transformation consisting of a boost v_x in the x-direction followed by a boost v_y in the y-direction. Show that the boosts performed in the reverse order would give a different transformation.

Problem 4 If two frames move with 3-velocities $\underline{V}_1$ and $\underline{V}_2$, show that their relative velocity is given by:

$$V^2 = \frac{(\underline{V}_1 - \underline{V}_2)^2 - (\underline{V}_1 \times \underline{V}_2)^2}{(1 - \underline{V}_1 \cdot \underline{V}_2)^2}$$

Problem 5 Frame S' moves with velocity $\underline{\beta}$ relative to frame S. A bullet in frame S' is fired with velocity $\underline{v}$ at an angel θ' with respect to the forward direction of motion. What is the angel θ as measured in S? What if the bullet is a photon?

Problem 6 Two successive, arbitrary pure Lorentz boosts $\underline{v}_1$ and $\underline{v}_2$ are equivalent to a pure boost $\underline{v}_3$ followed by a pure rotation $\theta\underline{n}$, where $\underline{n}$ is a unit vector. Find the magnitude of θ in terms of $\underline{v}_1$ and $\underline{v}_2$ and show that $\underline{n} \cdot \underline{v}_3 = 0$.

Problem 7 (Compton scattering) A photon of wavelength λ hits a stationary electron (mass m_e) and comes off with wavelength λ' at an angle θ. Derive the expression:

$$\lambda' - \lambda = (h/m_e)(1 - \cos\theta)$$

Problem 8

(a) When a photon scatters off a charged particle, which is moving with a speed very nearly that of light, the photon is said to have undergone an inverse Compton scattering. Consider an inverse Compton scattering in which a charged particle of rest mass m and total mass-energy (as seen in the lab frame) $E \gg m$, collides head-on with a photon of frequency ν ($h\nu \ll m$). What is the maximum energy the particle can transfer to the photon?

(b) If space is filled with black-body radiation of temperature 3°K and contains cosmic ray proton of energies up to 10^{20} eV, how much energy can a proton of energy 10^{20} eV transfer to a 3°K photon?

Problem 9

(a) If a rocket has engines that give it a constant acceleration of 1g (relative to its instantaneous inertial frame, of course), and the rocket starts from rest near the earth, how far from the earth (as measures in the earth's frame) will the rocket be in 40 years as measured on the earth? How far after 40 years as measured in the rocket?

(b) Compute the proper time for the occupants of a rocket ship to travel the 30, 000 light years from the Earth to the center of the galaxy. Assume they maintain as acceleration of 1g for half the trip and decelerate at 1g for the remaining half.

(c) What fraction of the initial mass of the rocket can be payload in part (b)? Assume an ideal rocket that converts rest mass into radiation and ejects all of the radiation out of the back with 100% efficiency and perfect collimation.

Problem 10 Find 4 linearly independent null vectors in Minikowski space. Can you find 4 which are orthogonal?

Problem 11 Let $A_{\mu\nu}$ be an antistmmetric tensor so that $A_{\mu\nu} = -A_{\nu\mu}$; and let $S^{\mu\nu}$ be symmetric tensor so that $S^{\mu\nu} = S^{\nu\mu}$. Show that $A_{\mu\nu} S^{\mu\nu} = 0$. Establish the following two identities for any arbitrary tensor $V_{\mu\nu}$. :

$$V^{\mu\nu} A_{\mu\nu} = 1/2\, (V^{\mu\nu} - V^{\nu\mu})\, A_{\mu\nu}, \quad V^{\mu\nu} S_{\mu\nu} = 1/2\, (V^{\mu\nu} + V^{\nu\mu})\, S_{\mu\nu}$$

Problem 12 What is the invariant volume element of contravariant momentum d^4P for 4-dimensional momentum space? What is the invariant 3-volume " on the mass shell," i.e. when the constraint $(-P\cdot P)^{1/2} = m$ is imposed?

Problem 13 For electric and magnetic fields, show that $B^2 - E^2$ and $\underline{E} \cdot \underline{B}$ are invariant under changes of coordinates and Lorentz transformations. Are there any invariants, which are not merely algebraic combinations of these two?

Problem 14 Prove that except when $(\underline{B}\cdot\underline{E})^2 + (B^2 - E^2)^2 = 0$, there is a Lorentz transformation, which will make $\underline{E}$ and $\underline{B}$ parallel ($\underline{E}' \times \underline{B}' = 0$). [**Hint:** Try $\underline{v} = a(\underline{E} \times \underline{B})$ for some a.]

Problem 15 Suppose that $\underline{E} \cdot \underline{B} = 0$. Show that there is a Lorentz transformation which makes $\underline{E} = 0$ if $B^2 - E^2 > 0$, or one that makes $B = 0$ if $B^2 - E^2 < 0$. What if $B^2 - E^2 = 0$ in addition to $\underline{E} \cdot \underline{B} = 0$?

Problem 16 Show by explicit examination of components that the equations:

$$F_{ij,k} + F_{jk,i} + F_{ki,j} = 0 \qquad\qquad F^{ij}{}_{,j} = 4\pi J^i$$

Reduce to Maxwell's equations:

$$\underline{\nabla} \cdot \underline{B} = 0,\ \ d\underline{B}/dt + \underline{\nabla} \times \underline{E} = 0,\ \ \underline{\nabla} \cdot \underline{E} = 4\pi\rho,\ \ d\underline{E}/dt - \underline{\nabla} \times \underline{B} = -4\pi \underline{J}$$

Problem 17

(a) Write out the $\mu = 0$ component of the Lorentz force equation $du^i/d\tau = (e/m)F^{ij}u_j$ expressing F^{ij} in terms of E_α and B_β, to obtain $dP^0/dt = e\underline{v} \cdot \underline{E}$.

(b) From the spatial components of the Lorentz 4-force equation, find an equation for $d\underline{P}/dt$ in terms of $\underline{E}$ and $\underline{B}$. (Here $\underline{P}$ is the spatial part of P).

Problem 18

(a) Show that the stress-energy tensor for the electromagnetic field is divergenceless (i.e. $T^{ij}{}_{,j} = 0$) in the absence of charge sources.

(b) Show that the stress-energy tensor for the electromagnetic field has zero trace ($T^i{}_i = o$).

Problem 19 The specific intensity I_ν of radiation measures the intensity of radiation at a particular frequency ν in a particular direction. It is defined as the flux per unit frequency interval, per unit solid angle. Show that I_ν/ν^3 is a Lorentz invariant.

Problem 20 A star emits radiation isotropically in its own rest frame, with (energy per unit time). At a particular instant, as measured from the earth the star is at a distance R, and is moving woth a velocity v which makes an angle θ with respect to the direction from the earth to the star. What is the flux of radiation (energy per unit area) seen by an observer on the earth in terms of R, v and θ evaluated at the instant the radiation was emitted?

Problem 21 Consider the Euclidean metric in spherical polar coordinates:

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2$$

(a) Compute explicitly the Christoffel symbols (feel free to use the computer if you have the software.

(b) Write out the 3 components of the geodesic equation (equation of motion for a free particle) and find the first integrals of the equation.

(c) For orbits in the equator ($\theta = \pi/2$), solve the equation to find $r(\phi)$.

(d) Show the straight lines satisfy explicitly the geodesic equation.

Problem 22

(a) Using the variational principle: $\delta \int g_{ij} (dx^i/d\lambda) (dx^j/d\lambda) d\lambda = 0$

derive the geodesic equation: $d^2x^a/d\lambda^2 + \Gamma^a_{ij} (dx^i/d\lambda) (dx^j/d\lambda) = 0$

where the Christoffel symbols $\Gamma^a_{ij} = g^{ab}(g_{bi,j} + g_{bj,i} - g_{ij,b})/2$

(b) Show explicitly that under a coordinate transformation x –> x', Γ^a_{ij}; do not transform as a tensor.

Problem 23

(a) Show explicitly that "radar distance" is given by:

$$dl^2 = (g_{\alpha\beta} + g_{o\alpha}\, g_{o\beta} /(-g_{oo}))\, dx^\alpha\, dx^\beta.$$

(c) Show that if observer A is "synchronized" with observer B and observer B is synchronized with observer C, observer A is not necessarily synchronized with observer C.

Problem 24 Show explicitly that:

(a) $A^i_{;i} = (A^i \sqrt{-g})_{,i} / \sqrt{-g}$ where g is the determinant of the metric tensor.

(b) $F^{ab}_{;b} = (\sqrt{-g}\, F^{AB})_{,B} / \sqrt{-g}$ for any anti-symmetric tensor F^{ab}.

(c) Write out all the terms of $F_{[abc]}$ where [ ] is the complete anti-symmetrization symbol. Simplify the terms when F_{abc} itself is already antisymmetric in the first two indices (a,b).

Problem 25 Suppose that the Maxwell tensor F_{ab} has the plane wave form:

$$F_{ab} = \lambda\, (p_a k_b - k_a p_b)$$

With $k_a k^a = 0 = p_a k^a$ and $p_a p^a = 1$ (p_a is the normalized space-like polarization vector).

(a) Write out the scalar $F_{ab}F^{ab}$ in terms of the E and B field.

(b) Using the above plane wave form show explicitly that $F_{ab}F^{ab} = 0$.

(c) Show that the stress-energy tensor $T_{ab} = \lambda^2 k_a k_b$ (λ^2 is related to the energy density of the plane wave).

(d) Compute $F_{ab;c}k^c$.

Problem 26 Prove explicitly the following properties of Lie derivatives:

(a) $L(aA + bB) = a\,L\,(A) + b\,L\,(B)$ if a,b are constants and A, B are tensors of the same rank.

(b) $L\,(A.B) = A\,L\,(B) + L\,(A).B$ where A,B are tensors of arbitrary ranks.

(c) If T is a tensor, then $L\,(T)$ transforms as a tensor.

(d) $L_A(B) = -\,L_B(A)$ for any 2 vectors A,B.

Problem 27 Show explicitly that the acceleration 4-vector a^i of the 1-g rocket of Prob. 9 is Fermi-transported along its world line, i.e. $D_F a/d\tau = 0$.

Problem 28

(a) If ξ is a Killing vector and u is tangent to a geodesic, show that $\xi^a u_a$ is a constant of motion, i.e. $D(\xi^a u_a)/d\lambda = 0$.

(b) In 2-dimenisoin Euclidean space $ds^2=dx^2+dy^2$, show explicitly that $\xi^a = (-y, x)$ is a Killing vector. What is ξ^a in polar coordinates?

Problem 29

(a) For the 2-dimensional sphere with metric $ds^2=d\theta^2 + \sin^2\theta\, d\phi^2$, find all independent Killing vectors.

(b) Let A^i have initial components = (1,0) at $\theta=\theta_o$, $\phi=0$ on the 2-sphere. What is A^i after it is parallel-transported around the latitude circle $\theta=\theta_o$?

Problem 30

(a) Verify explicitly the Bianchi Identities for the curvature tensor:(Hint: see text)

$$R_{abcd;k} + R_{abkc;d} + R_{abdk;c} = 0$$

(b) By contraction of the above identity, derive the contracted Bianchi Identity for the Ricci tensor:

$$(R^{ab} - g^{ab}R/2)_{;b} = 0$$

Problem 31 Consider the conformally flat metric:

$$g_{ab} = \eta_{ab}\exp(2\phi)$$

where η_{ab} is the Minkowski metric and ϕ any arbitrary scalar function.

(a) Compute explicitly all components of the curvature tensor R_{abcd}, the Ricci tensor R_{ab} and the Ricci scalar R.

(b) Show that the Weyl tensor $C_{abcd} = 0$ in this case

(c) If A_a (vector potential) is a solution of the vacuum Maxwell equations in Minkowski space, show that A_a is also a solution in the space defined by the above metric.(Hint: see textbook or Stephani)

Problem 32 For the equilibrium distribution function:

$$f_o = \frac{e^{x+{}_{\beta}aPa}}{1 + \varepsilon e^{x+{}_{\beta}aPa}}$$

$$\varepsilon = \pm 1, 0$$

Show that the particle flux $N^a = \int f_o p^a \pi$
And stress-energy tensor $T^{ab} = \int f_o p^a p^b \pi$
reduce to the perfect fluid forms. Express u^a in terms of β_a and n, p, m in terms of integrals of f_o

Problem 33

(a) If a space-like unit vector e^a is Fermi-Walker transported along a curve with tangent u^a, show that if e^a is orthogonal to u^a initially ($e_a u^a = 0$), it will always remain orthogonal to u^a.

(b) If two space-like unit vectors $e^a_{(1)}$ and $e^a_{(2)}$ are both Fermi-Walker transported along curve, show that if they are initially orthogonal to each other ($e^a_{(1)} e_{a(2)} = 0$), they will always remain orthogonal to each other. (Hint: show that the dot products are preserved by the Fermi-Walker transport)

Problem 34

(a) Show that the spatial components of $Dp^a/d\tau = eF_{ab}p^b/m$ reduces to the usual Lorentz force equation in flat spaces. What is the corresponding equation for the time component?

(b) If T^{ab} is the Maxwell stress-energy tensor. Showthat $T^{ab}{}_{;b} = -F^{ab}J_b$ where J_b is the 4-current.

Problem 35 Show explicitly that the coordinate transformation $u^2 - v^2 = (r/2M - 1)\exp(r/2M)$; $v/u = \tanh(t/4M)$ will bring the Schwarzschild metric into Kruskal form.
Find v(u) corresponding to r=2M and r=0.

Problem 36

(a) Express the gravitational redshift z of photons emitted by a stationary source at fixed r and received at infinity as functions of r

(b) Repeat above calculation for an emitter radially freely falling towards r=0 from infinity at rest.

(c) Show that the luminosity of a freely collapsing star seen at infinity decreases with t as $\exp(-t/M3^{3/2})$.

Problem 37

(a) Sketch the Schwarzschild potential for particles in circular orbit with constant angular momentum l: $V_{eff}=[(1-2M/r)(1+l^2/r^2)]^{1/2}$ for sample values of l=6M, 4M, 3.75M, $(12)^{1/2}$ M, and 2.5M

(b) Find the radii of circular orbits when l=4M and $=12^{1/2}$ M.
Are they stable or unstable?

(c) Find the gravitational red shifts of photons emitted at r=4M and 6M.

(d) Compute the binding energies of particles in Keplerian orbits at r=3M, 4M and 6M.

Problem 38

(a) Sketch the effective potential for photons: $V_{eff} = l^2(1-2M/r)/r^2$.

(b) Find the radius of the circular orbit when $dV_{eff}/dr = 0$. Is it stable or unstable?

(c) Determine the impact parameter b=l/k that corresponds to this orbit.

(d) Show that for photons with impact parameter $b >>> 3^{3/2}$ M it will be deflected by a total angle of $\Delta\phi=4M/b$.

Problem 39 Consider the diagonal Kasner metric $ds^2 = -dt^2 + a^2(t)dx^2 + b^2(t)dy^2+c^2(t)dz^2$ where a, b, c are different functions of t only.

(a) compute explicitly the extrinsic curvature $K_{\alpha\beta}$ (diagonal) and 3-space Ricci tensor ${}^{(3)}R_{\alpha\beta}$ (diagonal).

(b) For the vacuum Einstein equations $R_{ab}= 0$, show that the general solution has the form: $a = t^{p1}$, $b = t^{p2}$, $c = t^{p3}$. Where p_1, p_2, p_3 are constants.

(c) From the constraint equations show that p_1, p_2, p_3 satisfy the conditions $p_1+p_2+p_3 = 1 = p^2_1 + p^2_2 + p^2_3$.

(d) Show that p_1,p_2,p_3 can be parametrized by a single parameter $0\le s\le 1$:

$$p_1 = \frac{-s}{1+ s + s^2}, \qquad p_2 = \frac{s(1+s)}{1 + s +s^2} \qquad p_3 = \frac{1+s}{1 + s + s^2}$$

(e) Sketch p_1,p_2,p_3 as functions of s. What are the values of p when two of them are equal?

Problem 40 The geodesic equations of the Schwarzschild metric can be derived as the Lagrangian equation of the Lagrangian

$$L = \{e^{\lambda}(dr/d\tau)^2 + r^2(d\theta/d\tau)^2 +r^2\sin^2\theta(d\phi/d\tau)^2 - e^{\nu}(dt/d\tau)^2\}/2$$

(a) By comparing this with the form:

$$D^2x^i/d\tau^2 + \Gamma^i{}_{jk}\, dx^j/d\tau\, dx^k/d\tau = 0$$

Compute explicitly the Chrisfoffel symbols: $\Gamma^i{}_{jk}$

(b) Solve for the circular orbits at the equator $\theta=\pi/2$

(c) Show that for $r < 6M$ there is no stable circular orbit in a Schwarzschild metric.

Problem 41 For the Reissner-Nordstrom charged black hole, discuss the properties of the equatorial circular orbits ($\theta = \pi/2$) for the 3 separate cases $M< q$, $M = q$, $M>q$.

Are there marginally bound and marginally stable orbits as in the Schwarzschild case? Where is the photon orbit in the different cases? (assume the test particles are neutral).

Problem 42 For a constant density star $\mu = \mu_o$, show explicitly that the solution is indeed given by:

$$e^{-\lambda} = 1 - Ar^2, \; p + \mu = Be^{-\surd/2} \; , \; e^{\surd/2} = 4\pi G\, B/A \;\; - D\surd(1\text{-}Ar^2)$$

where A,B, D are constants. What is the gravitational redshift z of a photon emitted out the center of star and received at infinity? (assuming star transparent) (HINT: See Stephani)

Problem 43

(a) For circular orbits in the equator ($\theta = \pi/2$) of a Kerr black hole, express their radii as functions of the particle specific angular momentum and energy E.

(b) What is r of the <u>minimal</u> stable circular orbit?

(c) What are l and E at the minimal stable orbit?

(d) What is r and angular momentum for the circular photon orbit? (Note that your answers depend on whether the photon is co-rotating or counter-rotating with respect to the hole)

Problem 44

(a) For an observer orbiting a Kerr hole at the equator ($\theta = \pi/2$), Let $\Omega = d\phi/dt$ be his "coordinate"angular velocity. Show that his Ω cannot be zero if he is inside the ergosphere, but outside the horizon r_+.

(b) If he lies in $r_- \leq r < r_+$, show that he cannot remain at constant radius r, independent of how large his acceleration.

(c) Show that Ω ----> $a/(2Mr_+)$ as he approaches r_+.

Problem 45

(a) 2 stars of masses M_1 and M_2 separated by R are in a circular orbit around each other. Due to gravitational radiation, R changes with time. Using the quadropole formula for gravitational radiation loss and Newtonian orbits, find R(t) as the orbit decays.

(b) For an aspherical object of moment of inertia I and spinning at angular frequency ω , show that the maximum GW power radiated is dimensionally given by $L_{gw}\sim I^2\omega^6$.

(c) If this object is a binary system of identical stars of mass M, show that $L_{gw}\sim(M/R)^5$ where R is oribital separation.

(d) A nonspherical $1M_\odot$ neutron star collapses to a minimum size of 3 GM/c^2 and bounces on a time scale of sound crossing time. If the sound speed is $=c/\sqrt{3}$ what is the maximum L_{gw} in orders of magnitude? If the GW duration is ~ bounce time what is the total energy radiated? How does this compare with the gravitational binding energy? If the neutron star is at a distance of 1 kpc = 3.10^{21}cm what is the magnitude of the dimensionless amplitude h of the wave?

Problem 46

(a) For the Friedmann Universe satisfying the equation:

$$(da/dt)^2 - 8\pi m/3a = -k \qquad (k = 0, \pm 1)$$

Show that a is given by the parametric solution: $a(\eta) \propto dt/d\eta$

$$
\begin{array}{lll}
a(\eta) & \sim \eta - \sin\eta & k = +1 \\
 & \sim \eta^3/6 & k = 0 \\
 & \sim \sinh\eta - \eta & k = -1
\end{array}
$$

(b) For a star collapsing from initial radius R_o at $\eta = -\pi$ towards the singularity R= 0 at $\eta = 0$, Find the values of η when (1) $R(\eta)$ crosses the horizon; (2) when the last photon escaping to infinity must be emitted at the center of star; (3) η at center of star such that a photon emitted will reach surface R when R hits the singularity R=0

(c) Show that the singularity is "continuous" across the stellar surface (i.e. both the interior metric and exterior metric blow up as same power laws of η as $\eta \rightarrow 0$ along the matching boundary).